Miscellaneous Substitutions

(These are just a couple of examples, there are many more substitutions specific to particular equations)

An equation of the form

$$y' + p(x)y = q(x)y^n$$

is a Bernoulli equation.

Notice if $n=0$, then $y' + p(x)y = q(x)$ is linear.

If $n=1$, then $y' + p(x)y = q(x)y$ is separable.

So if $n \neq 0,1$ then the substitution $v = y^{1-n}$ always transforms the Bernoulli equation into a linear equation in v.

The general case is left for homework, I will illustrate the case $n=2$.

If $n=2$, then $y' + p(x)y = q(x)y^2$.

Let $v = y^{1-n} = y^{1-2} = \frac{1}{y}$

$$\frac{dv}{dx} = \frac{-1}{y^2}\frac{dy}{dx}$$

Dividing the Bernoulli equation by y^2 yields –

$$\frac{1}{y^2}y' + p(x)\cdot\frac{1}{y} = q(x)$$

or $\quad -\frac{dv}{dx} + p(x)v = q(x)$

$\frac{dv}{dx} - p(x)v = -q(x)$ which is linear in v. Then there exists an integrating factor $\mu(x) = e^{-\int p(x)dx}$ and

$$\frac{d}{dx}(\mu v) = \mu\frac{dv}{dx} + v\frac{d\mu}{dx}$$

$$= \mu\frac{dv}{dx} - \mu v p$$

which is the LHS of

$$\mu\frac{dv}{dx} - \mu p v = -\mu q$$

So $\frac{d}{dx}(\mu v) = -\mu q$

$$\mu v = -\int \mu q\, dx$$

and $v = -\frac{1}{\mu}\int \mu q\, dx$

where $\mu = e^{-\int p dx}$

After integrating for v you then substitute $v = \frac{1}{y}$ and solve for y. The solution works the same for any n.

Ricatti equations –

An equation of the form

$$\frac{dy}{dx} = q_1(x) + q_2(x)y + q_3(x)y^2$$

is a Ricatti equation.

Notice that if $q_3(x) = 0$ then the equation is linear and can be solved with the use of an integrating factor. In general, assume $q_3(x) \neq 0$.

Then if $y_1(x)$ is any solution, then $y = y_1(x) + \frac{1}{v(x)}$ is the general solution where $v(x)$ satisfies the linear equation

$$\frac{dv}{dx} = -(q_2 + 2q_3y_1)v - q_3 \quad \text{so we can find } v(x).$$

Proof: $y = y_1 + \frac{1}{v}$

$$\frac{dy}{dx} = \frac{dy_1}{dx} - \frac{1}{v^2}\frac{dv}{dx}$$

and $\frac{dy_1}{dx} - \frac{1}{v^2}\frac{dv}{dx} = q_1 + q_2\{y_1 + \frac{1}{v}\} + q_3\{y_1 + \frac{1}{v}\}^2$

$$= q_1 + q_2y_1 + q_2 \cdot \frac{1}{v} + q_3y_1^2 + 2q_3y_1 \cdot \frac{1}{v} + q_3 \cdot \frac{1}{v^2}$$

But y_1 is a solution of the Ricatti equation so

$$\frac{dy_1}{dx} = q_1 + q_2y_1 + q_3y_1^2$$

and $-\frac{1}{v^2}\frac{dv}{dx} = q_2 \cdot \frac{1}{v} + 2q_3y_1 \cdot \frac{1}{v} + q_3 \cdot \frac{1}{v^2}$

or $\frac{dv}{dx} = -q_2v - 2q_3y_1v - q_3$

$= -(q_2 + 2q_3y_1)v - q_3$ which is a linear eqn satisfied by v.

* Note: The use of this method depends on your ability to find a solution y_1. This is hopefully a trivial solution you can find by inspection.

ex: Solve $y' = 1 + x^2 - 2xy + y^2$.

This is a Ricatti equation. You can verify that $y_1 = x$ is a solution.

Then we suppose that $y = x + \frac{1}{v} = y_1 + \frac{1}{v}$ where $v = v(x)$.

So $\frac{dy}{dx} = 1 - \frac{1}{v^2}\frac{dv}{dx}$ and

$$1 - \frac{1}{v^2}\frac{dv}{dx} = 1 + x^2 - 2x(x + \tfrac{1}{v}) + (x + \tfrac{1}{v})^2$$

$$= 1 + x^2 - 2x^2 - \frac{2x}{v} + x^2 + \frac{2x}{v} + \frac{1}{v^2}$$

$$-\frac{1}{v^2}\frac{dv}{dx} = \frac{1}{v^2}$$

and, finally, $v = -x + c = c - x$ where c = constant.

So $y = y_1 + \frac{1}{v} = x + \frac{1}{c - x}$ is the general solution.

You should verify this!